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A matrix-valued solution to Bochner's problem

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Abstract
We exhibit families of matrix-valued functions $F(m, t)$, $m = 0, 1, 2, \ldots, t$ real, which are eigenfunctions of a fixed differential operator in t and of a fixed (block) tridiagonal semiinfinite matrix. Thus we have nontrivial solutions of a matrix-valued version of Bochner's problem. These functions arise as matrix-valued spherical functions associated to the two-dimensional complex projective space $SU(3)/U(2)$. In the very special case of one-dimensional representations of $U(2)$ they give instances of Jacobi polynomials that feature among the (scalar-valued) solutions of the problem posed and solved by Bochner back in 1929. This very classical work can be considered as the first instance of the 'bispectral problem' of recent interest in several aspects of mathematical physics.

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1. Introduction

Back in 1929, Bochner [B] posed the problem of determining all families of scalar-valued orthogonal polynomials that are eigenfunctions of some arbitrary but fixed second-order differential operator. This problem was solved by Bochner in the original paper and has resurfaced in different clothing many times since. This is not the appropriate place to review all these developments, and we just give references to a few recent papers where the reader can find a detailed presentation of these results [GH, H, SVZ].

It is important to notice that certain problems like the 'bispectral problem' considered in [DG] can be seen as a purely continuous version of a broad extension of the original Bochner problem. In the original formulation, back in [B], the pair of operators required by the bispectral property are the second-order difference operator furnished by the three-term recursion relation satisfied by any family of orthogonal polynomials and the second-order differential operator explicitly asked for by Bochner. It is now clear how these conditions on the orders of the operators can be relaxed to yield a rich variety of situations. The case when one is dealing with differential operators of arbitrary order has been considered for instance in [W1,W2], as well as in [BHY]. There is another sense in which [DG] differs from the original problem of Bochner, and this is a shift of emphasis away from any particular eigenfunction and rather a concentration on two 'local operators' (acting on different variables) whose joint eigenspaces could have arbitrary dimensions. This leads to considerations of the rank of the corresponding vector bundles over some algebraic curve. In the context of Bochner this leads one to deal with doubly infinite matrices and to abandon the insistence on polynomials, a point well stressed in [GH]. From this point of view this paper follows rather closely the work started in [B]. For some new developments see also [HI].

Many of the topics discussed in these papers have made interesting contacts with areas as varied as integrable systems, random matrix theory, interpolation and approximation theory, problems of electrostatic equilibrium, extensions of the Huygens principle, representations of the Weyl algebra, Calogero–Moser systems etc. It appears very natural, and maybe even profitable, to revisit the question in the matrix-valued context, and this is the theme of this paper.

Returning to the question raised by Bochner it is clear, starting with the work that stretches from Cartan to Harish-Chandra, see [GV], that a natural place to find examples satisfying his conditions is in the theory of spherical functions for symmetric spaces of rank one. In the compact case this leads to Jacobi polynomials, one of the four families that feature in the full solution of Bochner's problem. In fact it only leads to special cases of these Jacobi polynomials, a fact to which we will return later.

Now we turn to a description of the results in the present paper following a few introductory comments.

As remarked at the end of the introduction to [DG] some of the features of the bispectral problem lead one to suspect some sort of harmonic analysis in the background, even when a group is not obviously around. At any rate many of the more elaborate examples are obtained by applying the Darboux process to such basic situations where a symmetric space is present, see [DG]. With this in mind it is not unreasonable to go back to the 'group situation' in search for examples in this matrix-valued setup.

The general theory of scalar-valued spherical functions of arbitrary type, associated with a pair (G, K) with G a locally compact group and K a compact subgroup, goes back to Godement and Harish-Chandra. In [T], attention is focused on the underlying matrix-valued spherical functions defined as a solution of an integral identity. These two notions are related by the operation of taking traces. This theory is also developed in [GV].

When G is a Lie group the general theory see [T, GV] gives for a fixed irreducible representation (π, V) of K a family of matrix-valued functions that are eigenfunctions of a system of differential operators defined on the Lie group G. These spherical functions in fact take values in the set of linear maps from V into itself.

In [GPT] one finds a detailed elaboration of this theory when the symmetric space G/K is the complex projective plane. In this case we have $G = SU(3)$ and $K = S(U(2) \times U(1))$. To the best of our knowledge no other instance of an explicit description of the irreducible spherical functions of any K-type is known when the subgroup K is non-Abelian and thus the functions involved are not scalar valued.

This is not the appropriate place to repeat the results in [GPT] and it suffices to say that for each irreducible representation of K one eventually gets (by a proper combination of several spherical functions corresponding to this representation) a family of matrix-valued functions $\tilde{H}(t, w)$, $0 < t < 1$, $w = 0, 1, 2, 3, \ldots$ such that the differential equation

$$D\tilde{H}(t, w)^T = \tilde{H}(t, w)^T \Lambda \tag{1}$$

is satisfied for a differential operator D whose coefficients depend on t (and not on w); Λ is a diagonal matrix with entries that depend on w (but not on t).

Many other properties of these functions $\tilde{H}(t, w)$ are discussed in [GPT], including orthogonality, etc, but we do not make any attempt here to summarize the results in that rather long paper.

The last section in [GPT] deals with what is called there a 'rather intriguing matrix valued' form of the bispectral property enjoyed by $\tilde{H}(t, w)$. This last section makes a convincing case that as functions of the spectral parameter w they satisfy a three-term recursion relation of the form

$$t\tilde{H}(t, w) = A_w\tilde{H}(t, w-1) + B_w\tilde{H}(t, w) + C_w\tilde{H}(t, w+1) \tag{2}$$

where A_w, B_w and C_w are matrices independent of t. In [GPT] one finds explicit formulas for the matrices A and C above, as well as the off-diagonal entries in B. Full details are given in the case when the dimension of the representation of K is three, i.e. one of the two parameters (ℓ, n) that determine the representations of K satisfies $\ell = 2$. In the special case when $\ell = 0$ and $n = 0$ [GPT] gives the classical results involving Jacobi polynomials.

The main result of this paper is to give explicit formulae for these matrices $A(w)$, $B(w)$ and $C(w)$ for the case of an arbitrary irreducible representation of K and more importantly to give a sketch of the way in which the tensor product of certain representations of G can be used to get these explicit formulas. Full details, including proofs of the statements in section 3, will be given in a separate publication.

Before we take up this issue in the next section one could remark that another possible route to finding matrix-valued examples of Bochner's problem, even in the simplest case where all operators involved are of order two, would be to take the emerging general theory of matrix-valued orthogonal polynomials, see for instance [DVA], and to search here for interesting special cases where a second-order differential operator would exist. We learned of such an attempt through the kindness of Ismail [I] (see also [D]). For completeness one should say that the family of polynomial functions $\tilde{H}(t, w)$ that we consider here does not satisfy all the conditions of the theory in [DVA].

The results in [GPT] and this paper suggest that it may be worthwhile revisiting the general problem of Bochner in a matrix-valued context. The results discussed here are, to the best of our knowledge, the first nontrivial families of examples. It is natural to inquire, as the referees have done, about the situation for the pair $(G, K) = (SU(n), S(U(n-1) \times U(1))$ as well as other such families. At this point this appears to be a very interesting challenge.

2. Recursion relation for spherical functions

The aim of this section is to obtain a recursion relation for the spherical functions associated with the pair $(G, K) = (SU(3), S(U(2) \times U(1)))$.

2.1. The Lie algebra of $SU(3)$

The Lie algebra of G is $\mathfrak{g} = \{X \in \mathfrak{gl}(3, \mathbb{C}) : X = -\overline{X}^t, \operatorname{tr} X = 0\}$. Its complexification is $\mathfrak{g}_{\mathbb{C}} = \mathfrak{sl}(3, \mathbb{C})$. The Lie algebra $\mathfrak{k}$ of K can be identified with $\mathfrak{u}(2)$ and its complexification $\mathfrak{k}_{\mathbb{C}}$ with $\mathfrak{gl}(2, \mathbb{C})$.

The following matrices form a basis of $\mathfrak{g}$:

$$H_1 = \begin{bmatrix} \mathrm{i} & 0 & 0 \\ 0 & -\mathrm{i} & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad H_2 = \begin{bmatrix} \mathrm{i} & 0 & 0 \\ 0 & \mathrm{i} & 0 \\ 0 & 0 & -2\mathrm{i} \end{bmatrix}$$

$$Y_1 = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad Y_2 = \begin{bmatrix} 0 & \mathrm{i} & 0 \\ \mathrm{i} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$
$$Y_3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} \qquad Y_4 = \begin{bmatrix} 0 & 0 & \mathrm{i} \\ 0 & 0 & 0 \\ \mathrm{i} & 0 & 0 \end{bmatrix}$$
$$Y_5 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \qquad Y_6 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \mathrm{i} \\ 0 & \mathrm{i} & 0 \end{bmatrix}.$$

Let $\mathfrak{h}$ be the Cartan subalgebra of $\mathfrak{g}_{\mathbb{C}}$ of all diagonal matrices. The corresponding root space structure is given by

$$X_\alpha = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad X_{-\alpha} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad H_\alpha = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$
$$X_\beta = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \qquad X_{-\beta} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \qquad H_\beta = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$
$$X_\gamma = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad X_{-\gamma} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \qquad H_\gamma = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

where

$$\alpha(x_1 E_{11} + x_2 E_{22} + x_3 E_{33}) = x_1 - x_2$$
$$\beta(x_1 E_{11} + x_2 E_{22} + x_3 E_{33}) = x_2 - x_3$$
$$\gamma(x_1 E_{11} + x_2 E_{22} + x_3 E_{33}) = x_1 - x_3.$$

We have

$$X_\alpha = \tfrac{1}{2}(Y_1 - \mathrm{i}Y_2) \qquad X_\beta = \tfrac{1}{2}(Y_5 - \mathrm{i}Y_6) \qquad X_\gamma = \tfrac{1}{2}(Y_3 - \mathrm{i}Y_4)$$
$$X_{-\alpha} = -\tfrac{1}{2}(Y_1 + \mathrm{i}Y_2) \qquad X_{-\beta} = -\tfrac{1}{2}(Y_5 + \mathrm{i}Y_6) \qquad X_{-\gamma} = -\tfrac{1}{2}(Y_3 + \mathrm{i}Y_4).$$

Let $Z = H_\alpha + 2H_\beta$, $\tilde{H}_1 = 2H_\alpha + H_\beta$ and $\tilde{H}_2 = H_\beta - H_\alpha$.

2.2. Recursion relation on G

We identify $GL(2,\mathbb{C})$ and $U(2)$ with the subgroups of all 2×2 matrices of $GL(3,\mathbb{C})$ and respectively of $U(3)$ in the following way:

$$GL(2,\mathbb{C}) \simeq \begin{pmatrix} GL(2,\mathbb{C}) & 0 \\ 0 & 1 \end{pmatrix} \qquad U(2) \simeq \begin{pmatrix} U(2) & 0 \\ 0 & 1 \end{pmatrix}.$$

The equivalence classes of finite-dimensional irreducible holomorphic representations of $GL(3,\mathbb{C})$ are parametrized by the 3-tuples of integers $m_1 \geqslant m_2 \geqslant m_3$. When we restrict the representation $m_1 \geqslant m_2 \geqslant m_3$ of $GL(3,\mathbb{C})$ to $GL(2,\mathbb{C})$ it decomposes as the direct sum of the representations $k_1 \geqslant k_2$ of $GL(2,\mathbb{C})$ such that $m_1 \geqslant k_1 \geqslant m_2 \geqslant k_2 \geqslant m_3$, all of these with multiplicity one.

The irreducible representation of G with highest weight $\lambda = p\lambda_1 + q\lambda_2$ can be realized as the restriction to G of the representation $\tau_{(m_1,m_2,m_3)}$ of $GL(3,\mathbb{C})$ with $p = m_1 - m_2$ and $q = m_2 - m_3$. Also $\tau_{(m_1,m_2,m_3)}$ restricted to $U(2)$ is the direct sum of the representations $\tau_{(k_1,k_2)}$ with $m_1 \geqslant k_1 \geqslant m_2 \geqslant k_2 \geqslant m_3$. Moreover the irreducible $U(2)$-submodule V_{k_1,k_2} of V_{m_1,m_2,m_3} is an irreducible K-module of type (n,ℓ) with

$$\ell = k_1 - k_2 \qquad n = k_1 + 2k_2 - m_1 - m_2 - m_3.$$

The irreducible $GL(3,\mathbb{C})$-modulo $W=\mathbb{C}^3$ with the canonical action corresponds to the parameters $(1,0,0)$.

The following lemma gives the decomposition into irreducible $GL(3,\mathbb{C})$-modules of the tensor product of $V=V_{m_1,m_2,m_3}$ with W.

Lemma 2.1. *We have*

$$V\otimes W\simeq V^{\sigma_1}\oplus V^{\sigma_2}\oplus V^{\sigma_3}$$

where V^{σ_1}, V^{σ_2} and V^{σ_3} are irreducible $GL(3,\mathbb{C})$-modules of parameters

$$\sigma_1=(m_1+1,m_2,m_3)\qquad \sigma_2=(m_1,m_2+1,m_3)\qquad \sigma_3=(m_1,m_2,m_3+1).$$

Proof. See [Ze, p 227, theorem 2]. □

Now we take $V_1=V_{k_1,k_2}$ an irreducible $U(2)$-submodule of V_{m_1,m_2,m_3} and W_1 the irreducible $U(2)$-submodule of W of unit dimension, i.e. $W_1=\mathbb{C}e_3$. We note that $W_1=V_{0,0}$ as a K-module is of type $(-1,0)$. There exists a basis $\{v_k : k_1\geqslant k\geqslant k_2\}$ of V_1, taken from a Gelfand–Cetlin basis of V_{m_1,m_2,m_3}, of weight vectors v_k parametrized by the triangles

$$\begin{matrix} m_1 & & m_2 & & m_3 \\ & k_1 & & k_2 & \\ & & k & & \end{matrix}\;.$$

The weight of v_k is given by

$$kx_1+(k_1+k_2-k)x_2+(m_1+m_2+m_3-k_1-k_2)x_3.$$

The tensor product $V_1\otimes W_1$ is an irreducible $U(2)$-module of parameters $(k_1,k_2)+(0,0)=(k_1,k_2)$. The $GL(3,\mathbb{C})$-projection $P_j : V\otimes W\longrightarrow V^{\sigma_j}$ (for $j=1,2,3$) maps $V_1\otimes W_1$ onto the trivial module or onto the $U(2)$-submodule $V^{\sigma_j}_{k_1,k_2}$ of V^{σ_j}. For any v_k in the basis of V_1 we have

$$v_k\otimes e_3=v_1+v_2+v_3\ \in V^{\sigma_1}\oplus V^{\sigma_2}\oplus V^{\sigma_3}.$$

The vectors v_j are weight vectors in V^{σ_j} and belong to the $U(2)$-submodule $V^{\sigma_j}_{k_1,k_2}$. Thus the corresponding triangles of v_1, v_2, v_3 are, respectively,

$$\begin{matrix} m_1+1 & & m_2 & & m_3 \\ & k_1 & & k_2 & \\ & & k & & \end{matrix}\qquad \begin{matrix} m_1 & & m_2+1 & & m_3 \\ & k_1 & & k_2 & \\ & & k & & \end{matrix}\qquad \begin{matrix} m_1 & & m_2 & & m_3+1 \\ & k_1 & & k_2 & \\ & & k & & \end{matrix}\;.$$

We note that the vector $v_k\otimes e_3$ is of weight $(k, k_1+k_2-k, m_1+m_2+m_3+1-k_1-k_2)$ and each $V^{\sigma_j}_{k_1,k_2}$ is an irreducible K-module of type $(k_1+2k_2-m_1-m_2-m_3-1,\ell)=(n-1,\ell)$.

It is well known (see [Hu, p 32]) that there exists a basis $\{v_i\}_{i=0}^{\ell}$ of V_1 such that

$$\begin{aligned} \dot\pi(H_\alpha)v_i&=(\ell-2i)v_i\\ \dot\pi(X_\alpha)v_i&=(\ell-i+1)v_{i-1}\qquad (v_{-1}=0)\\ \dot\pi(X_{-\alpha})v_i&=(i+1)v_{i+1}\qquad (v_{\ell+1}=0).\end{aligned}\tag{3}$$

Lemma 2.2. *Let us consider a $U(2)$-invariant inner product on V_1. Then the basis $\{v_i\}_{i=0}^{\ell}$ described above is an orthogonal basis such that*

$$\|v_i\|^2=\binom{\ell}{i}\|v_0\|^2.$$

Proof. We have $\dot\pi(H_\alpha)^* = -\mathrm{i}\dot\pi(H_1)^* = \mathrm{i}\dot\pi(H_1)^* = \dot\pi(H_\alpha)$ and

$$\dot\pi(X_{-\alpha})^* = -\tfrac{1}{2}\big(\dot\pi(Y_1) + \mathrm{i}\dot\pi(Y_2)\big)^* = -\tfrac{1}{2}\big(-\dot\pi(Y_1) + \mathrm{i}\dot\pi(Y_2)\big) = \dot\pi(X_\alpha)$$

because $\dot\pi(Y)^* = -\dot\pi(Y)$ for all $Y \in \mathfrak{g}$.

Since $\dot\pi(H_\alpha)^* = \dot\pi(H_\alpha)$ and the v_i are eigenvectors corresponding to different eigenvalues of $\dot\pi(H_\alpha)$ they are orthogonal to each other.

Now the proof will be completed by induction on $0 \leqslant i \leqslant \ell$. The statement is clearly true for $i = 0$. Let us assume that the assertion is true for some $0 \leqslant i \leqslant \ell - 1$. Then

$$(i+1)\langle v_{i+1}, v_{i+1}\rangle = \langle \dot\pi(X_{-\alpha})v_i, v_{i+1}\rangle = \langle v_i, \dot\pi(X_\alpha)v_{i+1}\rangle = (\ell - i)\langle v_0, v_0\rangle.$$

Thus

$$\langle v_{i+1}, v_{i+1}\rangle = \frac{\ell - i}{i+1}\binom{\ell}{i}\langle v_0, v_0\rangle = \binom{\ell}{i+1}\langle v_0, v_0\rangle.$$

□

Proposition 2.3. *Let $\{v_i\}_{i=0}^{\ell}$ be a basis of V_1 such that (3) holds, and equip V with a G-invariant inner product. Similarly take on W the G-invariant inner product such that $\|e_3\| = 1$. Let a_j be defined by*

$$v_0 \otimes e_3 = a_1 v_0^{\sigma_1} + a_2 v_0^{\sigma_2} + a_3 v_0^{\sigma_3} \in V^{\sigma_1} \oplus V^{\sigma_2} \oplus V^{\sigma_3} \tag{4}$$

with $a_j > 0$ and $\|v_0^{\sigma_j}\| = 1$. Let $v_i^{\sigma_j} \in V^{\sigma_j}$ be defined by

$$v_i \otimes e_3 = a_1 v_i^{\sigma_1} + a_2 v_i^{\sigma_2} + a_3 v_i^{\sigma_3}.$$

Then $\{v_i^{\sigma_j}\}_{i=0}^{\ell}$ ($j = 1, 2, 3$) is a basis of an irreducible $U(2)$-module $V_1^{\sigma_j}$ contained in V^{σ_j} such that (3) holds. If $P_j(v_0 \otimes e_3) = 0$ we take $a_j = 0$ and we do not define $v_i^{\sigma_j}$. Hence

$$\|v_i^{\sigma_j}\|^2 = \binom{\ell}{i}.$$

Proof. Since P_j is in particular a $U(2)$-morphism and e_3 is a $U(2)$-invariant, from (4) it follows that each $v_0^{\sigma_j}$ is a $U(2)$-dominant vector of weight ℓ.

On the other hand we have

$$\begin{aligned} a_1 X_{-\alpha}^i(v_0^{\sigma_1}) + a_2 X_{-\alpha}^i(v_0^{\sigma_2}) + a_3 X_{-\alpha}^i(v_0^{\sigma_3}) &= X_{-\alpha}^i(v_0 \otimes e_3) = i!\, v_i \otimes e_3 \\ &= i!\,\big(a_1 v_i^{\sigma_1} + a_2 v_i^{\sigma_2} + a_3 v_i^{\sigma_3}\big). \end{aligned}$$

Therefore $X_{-\alpha}^i(v_0^{\sigma_j}) = i!\, v_i^{\sigma_j}$ for $j = 1, 2, 3$. This completes the proof of the proposition. □

Theorem 2.4. *Let Φ be the spherical function of type (n, ℓ) associated with the G-module V. Let ϕ be the spherical function of type $(-1, 0)$ associated with the G-module W. Let Φ^{σ_j} be the spherical function of type $(n-1, \ell)$ associated with the G-module V^{σ_j} ($j = 1, 2, 3$). Then*

$$\Phi(g)\,\phi(g) = a_1^2\,\Phi^{\sigma_1}(g) + a_2^2\,\Phi^{\sigma_2}(g) + a_3^2\,\Phi^{\sigma_3}(g).$$

Proof. Let $u_i = \binom{\ell}{i}^{-1/2} v_i$ and let $u_i^{\sigma_j} = \binom{\ell}{i}^{-1/2} v_i^{\sigma_j}$. Then $\{u_i\}_0^{\ell}$ and $\{u_i^{\sigma_j}\}_0^{\ell}$ are, respectively, orthonormal bases of V_1 and $V_1^{\sigma_j}$ for $j = 1, 2, 3$. On the one hand we have

$$\langle g(u_j \otimes e_3), u_i \otimes e_3\rangle = \langle g u_j, u_i\rangle\,\langle g e_3, e_3\rangle.$$

On the other hand we get

$$\begin{aligned} \langle g(u_j \otimes e_3), u_i \otimes e_3\rangle &= \langle a_1 g u_j^{\sigma_1} + a_2 g u_j^{\sigma_2} + a_3 g u_j^{\sigma_3}, a_1 u_i^{\sigma_1} + a_2 u_i^{\sigma_2} + a_3 u_i^{\sigma_3}\rangle \\ &= a_1^2\langle g u_j^{\sigma_1}, u_i^{\sigma_1}\rangle + a_2^2\langle g u_j^{\sigma_2}, u_i^{\sigma_2}\rangle + a_3^2\langle g u_j^{\sigma_3}, u_i^{\sigma_3}\rangle. \end{aligned}$$

Therefore

$$\Phi_{ij}(g)\phi(g) = a_1^2\,\Phi_{ij}^{\sigma_1}(g) + a_2^2\,\Phi_{ij}^{\sigma_2}(g) + a_3^2\,\Phi_{ij}^{\sigma_3}(g).$$

This completes the proof of the theorem. □

2.3. Reduction to one variable

Now we want to express the identity in theorem 2.4 in terms of the functions $H(t)$, associated with the spherical functions, given in sections 10 and 11 of [GPT]. For details and definitions see [GPT].

For any $s \in \mathbb{R}$ let

$$a(s) = \begin{pmatrix} \cos s & 0 & \sin s \\ 0 & 1 & 0 \\ -\sin s & 0 & \cos s \end{pmatrix}.$$

If we put $A(s) = \begin{pmatrix} \cos s & 0 \\ 0 & 1 \end{pmatrix}$, for $-\pi/2 < s < \pi/2$, and Φ denotes a spherical function of type (n, ℓ), we have

$$\Phi(a(s)) = (\cos s)^n H(a(s)) A(s)^\ell = (\cos s)^n \tilde{H}(\tan s) A(s)^\ell$$

since $p(a(s)) = (\tan s, 0, 1)$.

If we make the change of variables $t = \cos^2 s$ we have

$$\Phi(a(s)) = t^{\frac{n}{2}} H(t) \begin{pmatrix} t^{\frac{1}{2}} & 0 \\ 0 & 1 \end{pmatrix}^\ell$$

where the exponent ℓ denotes the ℓth symmetric power of the matrix.

The spherical function $\phi(g)$ of type $(-1, 0)$ associated with the G-module W satisfies $\phi(a(s)) = (\cos s)^{-1} h(\tan s)$, and a direct computation gives $\phi(a(s)) = \cos s$. Thus the associated function h in the variable t is $h(t) = t$.

Corollary 2.5. *Let $H = H(t)$ be the function corresponding to the spherical function of type (n, ℓ) associated with the G-module V. Let $H^{\sigma_j} = H^{\sigma_j}(t)$ be the function corresponding to the spherical function of type $(n-1, \ell)$ associated with the G-module V^{σ_j}, $j = 1, 2, 3$. Then*

$$t\, H(t) = a_1^2 H^{\sigma_1}(t) + a_2^2 H^{\sigma_2}(t) + a_3^2 H^{\sigma_3}(t).$$

Proof. From theorem 2.4 we get

$$\Phi(a(s))\, \phi(a(s)) = a_1^2\, \Phi^{\sigma_1}(a(s)) + a_2^2\, \Phi^{\sigma_2}(a(s)) + a_3^2\, \Phi^{\sigma_3}(a(s)).$$

Making the change of variables $t = \cos^2 s$ we obtain

$$t^{\frac{n+1}{2}} H(t) \begin{pmatrix} t^{\frac{1}{2}} & 0 \\ 0 & 1 \end{pmatrix}^\ell = \big(a_1^2 H^{\sigma_1}(t) + a_2^2 H^{\sigma_2}(t) + a_3^2 H^{\sigma_3}(t)\big) t^{\frac{n-1}{2}} \begin{pmatrix} t^{\frac{1}{2}} & 0 \\ 0 & 1 \end{pmatrix}^\ell.$$

Since for $t \neq 0$ the matrix $\begin{pmatrix} t^{\frac{1}{2}} & 0 \\ 0 & 1 \end{pmatrix}^\ell$ is nonsingular, the corollary follows. □

Now we want to reformulate corollary 2.5 in terms of the parameters n, ℓ, k, w, introduced in section 9 of [GPT]. First of all we need the following lemma.

Lemma 2.6. *Let $V = V_{m_1,m_2,m_3}$ be a $GL(3, \mathbb{C})$ irreducible module and let V_1 be a $U(2)$ irreducible submodule of V of parameters k_1, k_2. Now let us consider the $GL(3, \mathbb{C})$ irreducible module $U = U_{m_1-m_3,m_2-m_3,0}$ and the corresponding $U(2)$ irreducible submodule U_1 of U of parameters $k_1 - m_3, k_2 - m_3$. Then the spherical functions Φ_{V,V_1} and Φ_{U,U_1} associated with the G-modules V and U and the K-submodules V_1 and U_1, respectively, are equivalent.*

Proof. It is clear that the G-modules V and U are equivalent. Moreover V_1 as a K-submodule of V is of type (n, ℓ) with $n = k_1 + 2k_2 - m_1 - m_2 - m_3$ and $\ell = k_1 - k_2$. Thus the corresponding $U(2)$-submodule U_1 of U must have parameters k_1', k_2' such that $n = k_1' + 2k_2' - (m_1 - m_3) - (m_2 - m_3)$ and $\ell = k_1' - k_2'$. From this it follows easily that $k_1' = k_1 - m_3$ and that $k_2' = k_2 - m_3$. □

Corollary 2.7. *Let $H = H(n, \ell, k, w; t)$ be the function corresponding to the spherical function on G of type (n, ℓ) associated with the parameters k, w. Then*

$$t\,H(n, \ell, k, w; t) = a_1^2 H(n-1, \ell, k, w+1; t) + a_2^2 H(n-1, \ell, k+1, w; t) + a_3^2 H(n-1, \ell, k, w; t).$$

We recall that the parameters n, ℓ, k, w are integers subject to the conditions $0 \leqslant w$, $0 \leqslant k \leqslant \ell$ and $0 \leqslant w + n + k$. We also note that the constants a_j depend on n, ℓ, k, w but not on t.

Proof. To identify the spherical functions appearing in the statement of theorem 2.4 we may assume that $m_3 = 0$. Then we have the following relations (section 9 of [GPT]): $p = m_1 - m_2$, $q = m_2$, $n = k_1 + 2k_2 - p - 2q$, $\ell = k_1 - k_2$, $w = p + q - k_1$ and $k = q - k_2$. Then $\Phi(g) = \Phi(n, \ell, k, w; g)$, $\phi(g) = \phi(-1, 0, 1, 0; g)$, $\Phi^{\sigma_1}(g) = \Phi(n-1, \ell, k, w+1; g)$ and $\Phi^{\sigma_2}(g) = \Phi(n-1, \ell, k+1, w; g)$. To identify Φ^{σ_3} one first uses lemma 2.6 and then one computes the new parameters and obtains $\Phi^{\sigma_3}(g) = \Phi(n-1, \ell, k, w; g)$. This completes the proof of the corollary. □

3. The bispectral property

Let W^* denote the $GL(3, \mathbb{C})$-module dual to W. Then $W^* = (0, 0, -1)$. Replacing W in the previous section by W^* we obtain the following results.

Lemma 3.1. *If $V = (m_1, m_2, m_3)$ then*

$$V \otimes W^* \simeq V^{\tau_1} \oplus V^{\tau_2} \oplus V^{\tau_3}$$

where V^{τ_1}, V^{τ_2} and V^{τ_3} are irreducible $GL(3, \mathbb{C})$-modules of parameters

$$\tau_1 = (m_1 - 1, m_2, m_3) \qquad \tau_2 = (m_1, m_2 - 1, m_3) \qquad \tau_3 = (m_1, m_2, m_3 - 1).$$

Now we take $W_1^* = \mathbb{C}e_3^*$. Then $W_1^* = (0, 0)$ as a $U(2)$-submodule of W^*, and as a K-module is of type $(1, 0)$.

Let V_1 be the $U(2)$-submodule of $V = (m_1, m_2, m_3)$ of parameters (k_1, k_2). Then the $GL(3, \mathbb{C})$-projection $P_j : V \otimes W^* \longrightarrow V^{\tau_j}$ (for $j = 1, 2, 3$) maps $V_1 \otimes W_1^*$ onto the trivial module or onto the $U(2)$-submodule $V_{k_1,k_2}^{\tau_j}$ of V^{τ_j}. Observe that $V_{k_1,k_2}^{\tau_j}$ as a K-module is of type $(n+1, \ell)$.

Theorem 3.2. *Let Φ be the spherical function of type (n, ℓ) associated with the G-module V. Let ψ be the spherical function of type $(1, 0)$ associated with the G-module W^*. Let Φ^{τ_j} be the spherical function of type $(n+1, \ell)$ associated with the G-module V^{τ_j}, ($j = 1, 2, 3$). Then*

$$\Phi(g)\,\psi(g) = b_1^2\,\Phi^{\tau_1}(g) + b_2^2\,\Phi^{\tau_2}(g) + b_3^2\,\Phi^{\tau_3}(g).$$

Corollary 3.3. *Let $H = H(t)$ be the function corresponding to the spherical function of type (n, ℓ) associated with the G-module V. Let $H^{\tau_j} = H^{\tau_j}(t)$ be the function corresponding to the spherical function of type $(n+1, \ell)$ associated with the G-module V^{τ_j}, $j = 1, 2, 3$. Then*

$$H(t) = b_1^2 H^{\tau_1}(t) + b_2^2 H^{\tau_2}(t) + b_3^2 H^{\tau_3}(t).$$

Corollary 3.4. *Let $H = H(n, \ell, k, w; t)$ be the function corresponding to the spherical function on G of type (n, ℓ) associated with the parameters k, w. Then*

$$H(n-1, \ell, k, w; t) = b_1^2 H(n, \ell, k, w-1; t) + b_2^2 H(n, \ell, k-1, w; t) + b_3^2 H(n, \ell, k, w; t).$$

The constants b_j depend on n, ℓ, k, w but not on t.

If we combine the results of corollaries 2.7 and 3.4 we obtain:

Proposition 3.5. *If we fix the K-type (n,ℓ) we may write $a_j(n,\ell,k,w) = a_j(k,w)$ and $b_j(n,\ell,k,w) = b_j(k,w)$ for $j = 1,2,3$. In the following equation we also write $a_j = a_j(k,w)$. Then we have*

$$\begin{aligned} t\,H(n,\ell,k,w;t) = {} & (a_1^2 b_1^2(k,w+1) + a_2^2 b_2^2(k+1,w) + a_3^2 b_3^2(k,w))H(k,w;t) \\ & + a_1^2 b_2^2(k,w+1)H(k-1,w+1;t) + a_1^2 b_3^2(k,w+1)H(k,w+1;t) \\ & + a_2^2 b_1^2(k+1,w)H(k+1,w-1;t) + a_2^2 b_3^2(k+1,w)H(k+1,w;t) \\ & + a_3^2 b_1^2(k,w)H(k,w-1;t) + a_3^2 b_2^2(k,w)H(k-1,w;t). \end{aligned}$$

Proposition 3.6. *If we fix the K-type $(n-1,\ell)$ we may write $a_j(k,w) = a_j(n,\ell,k,w)$ and $b_j(k,w) = b_j(n,\ell,k,w)$ for $j = 1,2,3$. In the following equation we also write $b_j = b_j(k,w)$; then we have*

$$\begin{aligned} t\,H(n-1,\ell,k,w;t) = {} & (b_1^2 a_1^2(k,w-1) + b_2^2 a_2^2(k-1,w) + b_3^2 a_3^2(k,w))H(k,w;t) \\ & + b_1^2 a_2^2(k,w-1)H(k+1,w-1;t) + b_1^2 a_3^2(k,w-1)H(k,w-1;t) \\ & + b_2^2 a_1^2(k-1,w)H(k-1,w+1;t) + b_2^2 a_3^2(k-1,w)H(k-1,w;t) \\ & + b_3^2 a_1^2(k,w)H(k,w+1;t) + b_3^2 a_2^2(k,w)H(k+1,w;t). \end{aligned}$$

For given integers $\ell \geqslant 0$, $w \geqslant 0$ and n, consider the matrix whose rows are given by the vectors $H(n,\ell,k,w;t)$ corresponding to the values $k = 0,1,2,\ldots,\ell$. Denote the corresponding matrix by $\tilde{H}(n,w;t)$. As a function of t, $\tilde{H}(n,w;t)$ satisfies a second-order differential equation

$$\mathrm{D}\tilde{H}(n,w;t)^T = \tilde{H}(n,w,t)^T \Lambda.$$

Here Λ is a diagonal matrix with $\Lambda(i,i) = -w(w+n+i+\ell+2) - i(n+i+1)$, $0 \leqslant i \leqslant \ell$; D is the differential operator introduced in [GPT]. Moreover we have:

Theorem 3.7. *There exist matrices A_w, B_w and C_w independent of t, such that*

$$t\tilde{H}(n,w;t) = A_w\tilde{H}(n,w-1;t) + B_w\tilde{H}(n,w;t) + C_w\tilde{H}(n,w+1;t).$$

More precisely we may take

$$A_w = \sum_{i=0}^{\ell} a_3^2(i,w)b_1^2(i,w)E_{i,i} + \sum_{i=0}^{\ell-1} a_2^2(i,w)b_1^2(i+1,w)E_{i,i+1}$$

$$\begin{aligned} B_w = {} & \sum_{i=0}^{\ell} \left(a_1^2(i,w)b_1^2(i,w+1) + a_2^2(i,w)b_2^2(i+1,w) + a_3^2(i,w)b_3^2(i,w)\right)E_{i,i} \\ & + \sum_{i=0}^{\ell-1} a_3^2(i,w)b_2^2(i,w)E_{i+1,i} + \sum_{i=0}^{\ell-1} a_2^2(i,w)b_3^2(i+1,w)E_{i,i+1} \end{aligned}$$

$$C_w = \sum_{i=0}^{\ell} a_1^2(i,w)b_3^2(i,w+1)E_{i,i} + \sum_{i=0}^{\ell-1} a_1^2(i,w)b_2^2(i,w+1)E_{i+1,i}.$$

Finally, the explicit values of the quantities a_j^2 and b_j^2 are given by

$$a_1^2(i,w) = \frac{(w+1)(w+\ell+2)}{(2w+n+\ell+i+2)(w+\ell-i+1)}$$

$$a_2^2(i,w) = \frac{(i+1)(\ell-i)}{(w+n+2i+1)(w+\ell-i+1)}$$

$$a_3^2(i,w) = \frac{(w+n+i)(w+n+\ell+i+1)}{(2w+n+\ell+i+2)(w+n+2i+1)}$$
$$b_1^2(i,w) = \frac{w(w+\ell+1)}{(2w+n+\ell+i+1)(w+\ell-i+1)}$$
$$b_2^2(i,w) = \frac{i(\ell-i+1)}{(w+n+2i)(w+\ell-i+1)}$$
$$b_3^2(i,w) = \frac{(w+n+i)(w+n+\ell+i+1)}{(2w+n+\ell+i+1)(w+n+2i)}.$$

The full derivation of these results will appear elsewhere.

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